Emmy Noether's Ascending Chain of Abstraction

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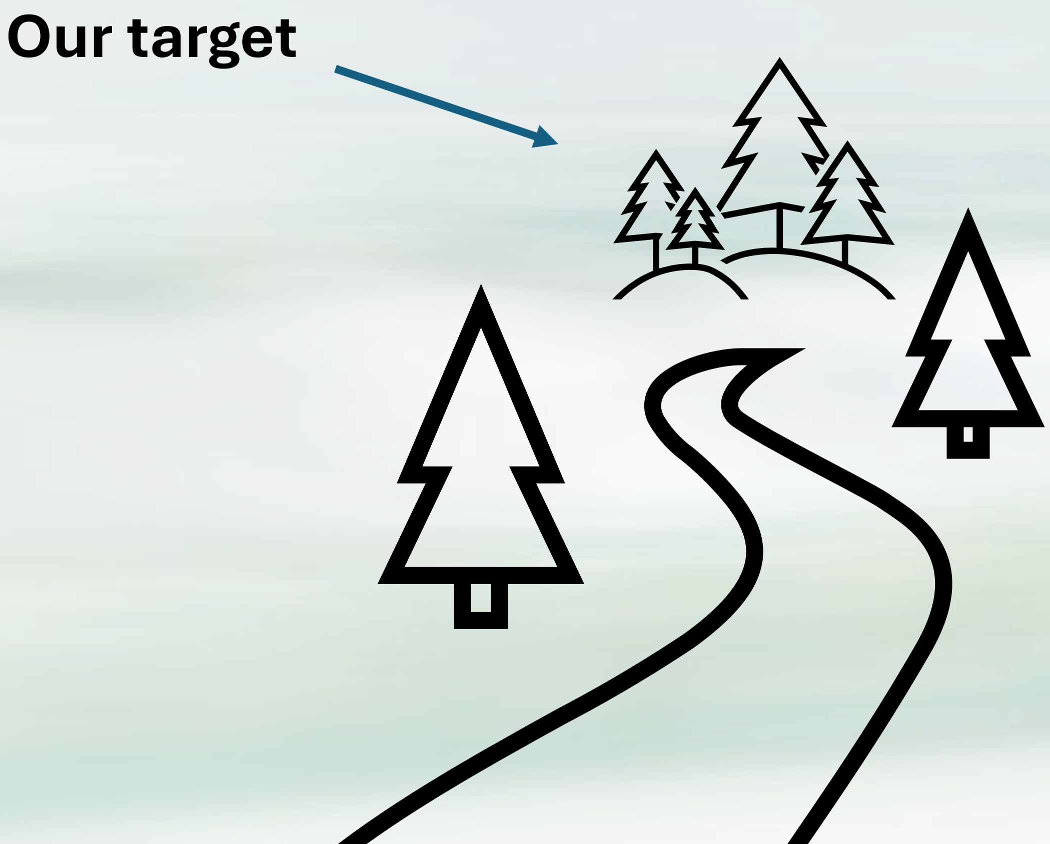
Early Challenges of Emmy Noether

- **1882** Childhood
- **1904** Enrols at Erlangen
- **1907** PhD on invariant theory
- **1915** Invited to Göttingen
- **1918** ‘Noether’s theorem’
- **1921** ‘Noetherian rings’

We are a university, not a bathing establishment.

Weyl: her strength was to strip away the inessentials

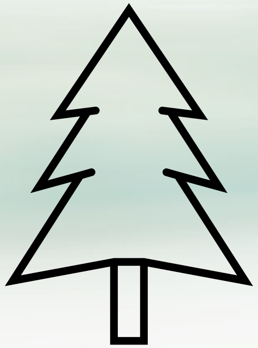
It's important to see the forest through the trees…



Let's start with the integers...

$$3 \mid x,\ 3 \mid y \implies 3 \mid (x - y)$$

$$3 \mid x \implies 3 \mid (xr), \text{for any } r \in \mathbb{Z}$$



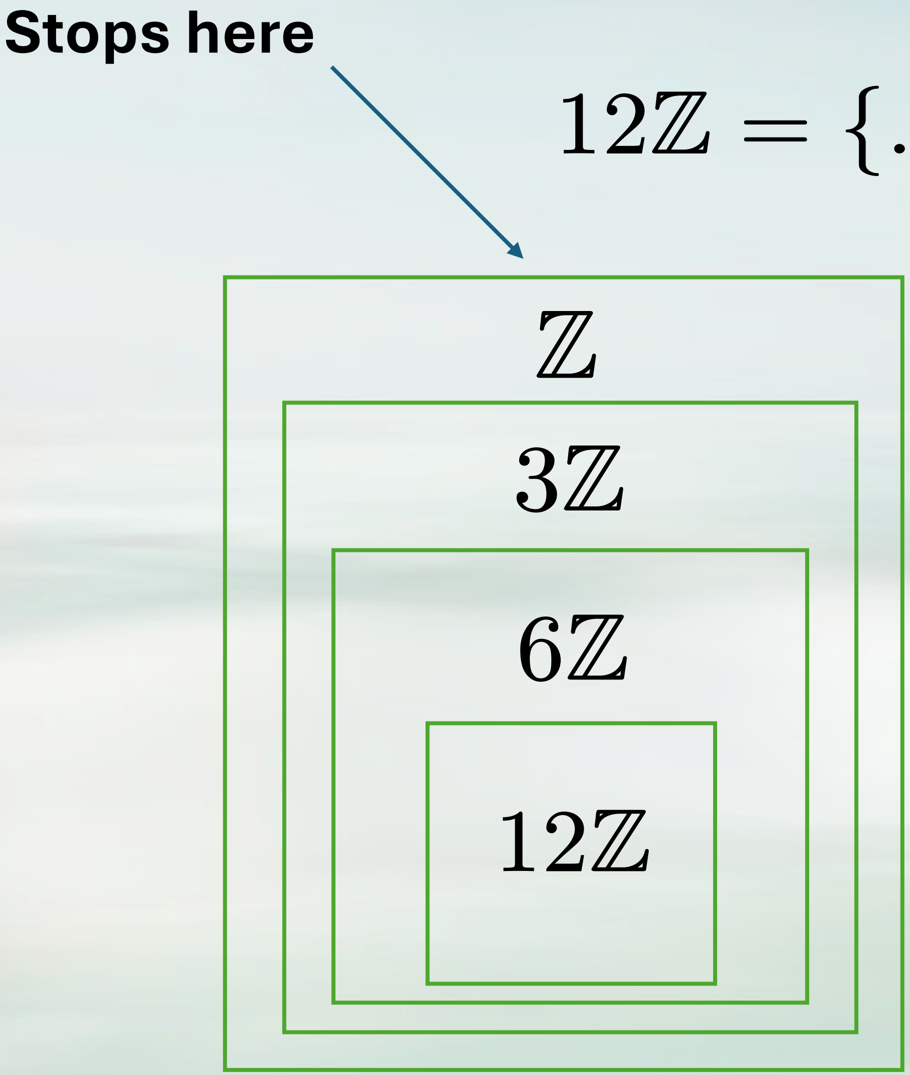
What about any arbitrary ring…

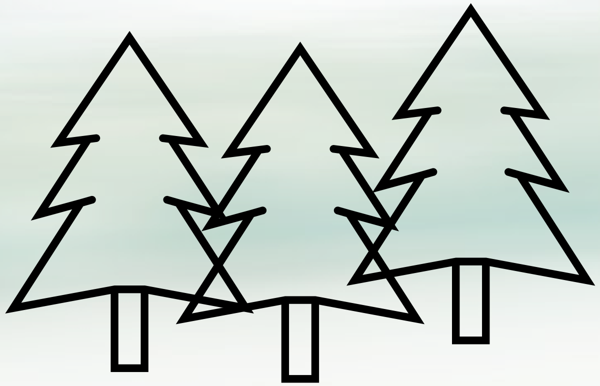
$$x, y \in I \;\Rightarrow\; x - y \in I$$

$$x \in I,\ r \in R \;\Rightarrow\; rx \in I$$

Ideals in the integers

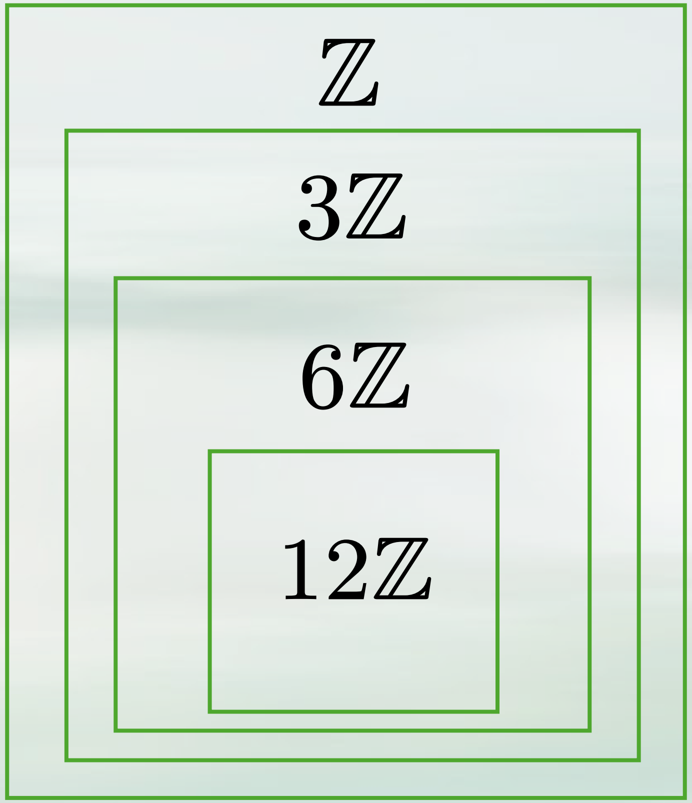
$$12\mathbb{Z} = \{\dots, -24, -12, 0, 12, 24, \dots\}$$



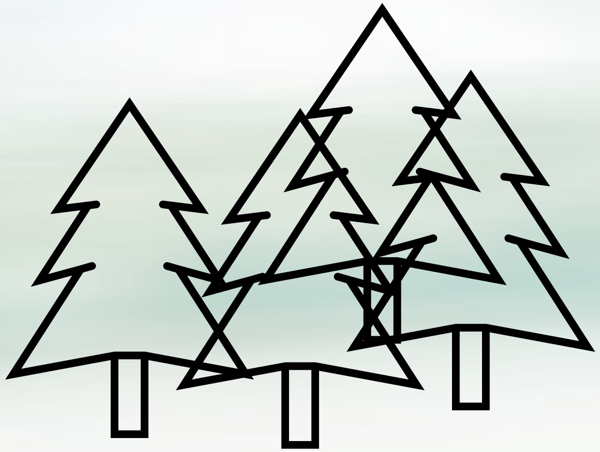


Ideals in the integers

$$12\mathbb{Z} \subseteq 6\mathbb{Z} \subseteq 3\mathbb{Z} \subseteq \mathbb{Z}$$



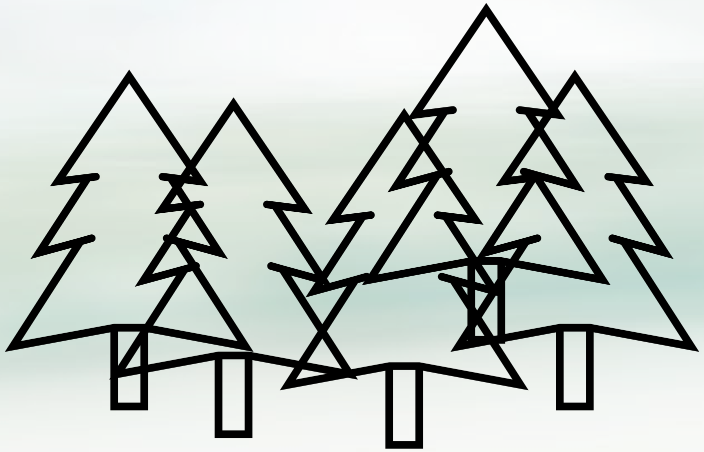
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
$$\implies \exists n \text{ with } I_n = I_{n+1} = I_{n+2} = \cdots$$



Polynomial rings

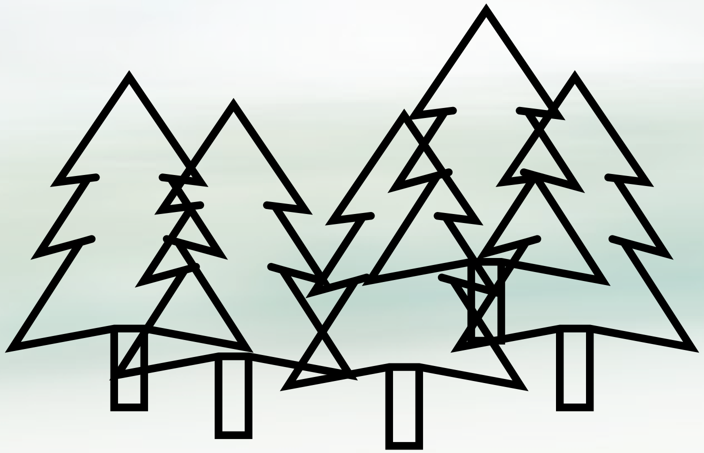
$$R[x] = \{ r_0 + r_1 x + r_2 x^2 + \cdots + r_n x^n \mid r_i \in R,\ n \geq 0 \}$$

If *R* is Noetherian, what happens when we add a variable?



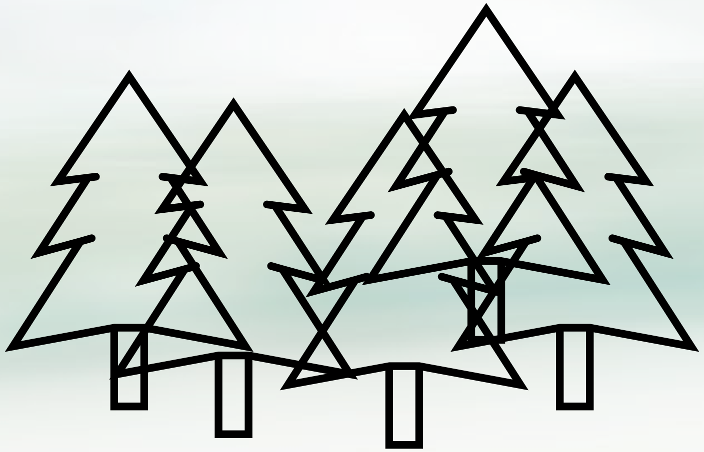
Hilbert's Basis Theorem

If R is Noetherian, then $R[x]$ is Noetherian.



The quest of the general skew polynomial ring

~~$xr = rx$~~



The quest of the general skew polynomial ring

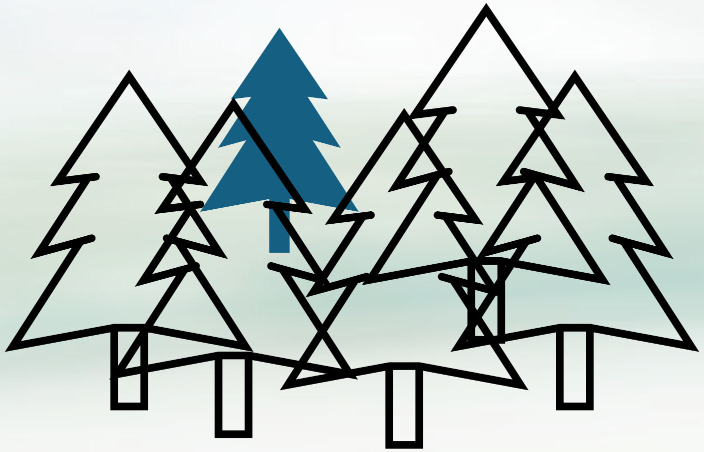
~~$$xr = rx$$~~

Skew polynomial rings – do they pass the test of the Basis Theorem

If R is Noetherian, then $R[x]$ is Noetherian.

$$xr = \alpha(r)x \quad \longrightarrow \quad R[x;\alpha] \quad ✓$$

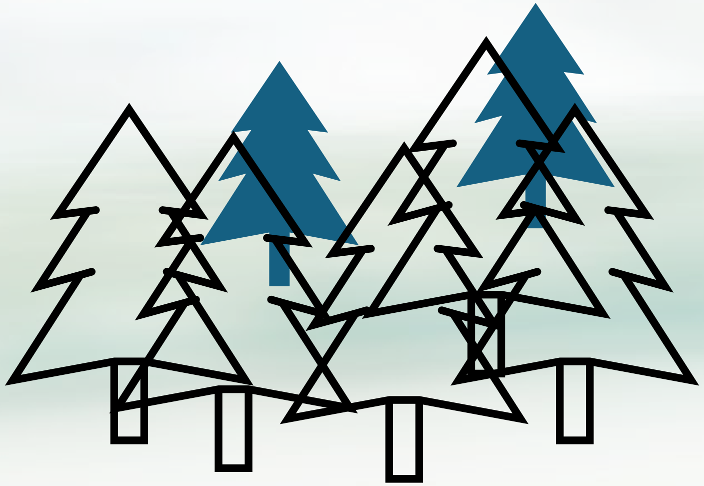
$$x^{-1}r = \alpha^{-1}(r)x^{-1} \quad \longrightarrow \quad R[x^{\pm 1};\alpha] \quad ✓$$



Formal differential operator rings– do they pass the test of the Basis Theorem

If R is Noetherian, then $R[x]$ is Noetherian.

$xr = rx + \delta(r)$ ⟶ $R[x;\delta]$ ✓



How to build a polynomial ring?

Recipe:

1. Polynomial

2. Ring

How to build a polynomial ring?

We need “polynomial”:

$$\sum a_i x^i = \sum b_i x^i \quad \Longleftrightarrow \quad a_i = b_i$$

$$\deg(xr) \leq \deg(x) + \deg(r)$$

How to build a polynomial ring?

We need "polynomial":

$$\deg(xr) \leq \deg(x) + \deg(r)$$

$$\deg(x) = 1 \qquad\qquad \deg(r) = 0$$

$$\deg(xr) \leq 1$$

$$xr = r'x + r''$$

How to build a polynomial ring?

We need “polynomial”:

$$xr = r'x + r''$$

$$xr = \alpha(r)x + \delta(r)$$

How to build a polynomial ring?

$$xr = \alpha(r)x + \delta(r)$$

We need "ring":

Multiplicative identity $r = 1 \implies \alpha(1) = 1 \text{ and } \delta(1) = 0$

Distributivity $\alpha(r+s) = \alpha(r) + \alpha(s) \text{ and } \delta(r+s) = \delta(r) + \delta(s)$

How to build a polynomial ring?

$$xr = \alpha(r)x + \delta(r)$$

We need "ring":

Associativity...

$$\alpha(rs)x + \delta(rs) = (\alpha(r)x + \delta(r))s = \alpha(r)(xs) + \delta(r)s = \alpha(r)(\alpha(s)x + \delta(s)) + \delta(r)s$$

$$\alpha(rs) = \alpha(r)\alpha(s)$$

$$\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$$

Shopping list for a polynomial ring

$$xr = \alpha(r)x + \delta(r)$$

$$\alpha(1) = 1 \qquad \delta(1) = 0$$

$$\alpha(rs) = \alpha(r)\alpha(s)$$

$$\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$$

Is this $R[x; \alpha, \delta]$ Noetherian?

Is this skew polynomial ring still Noetherian?

Yes, except…

It's important to see the name behind the theorems...

All this is thanks to Emmy Noether

Emmy Noether shaped algebra and physics. She worked for years without pay or recognition. Are the doors truly more open now?