TWISTS, TURNS AND SYMMETRY: THE RUBIK'S CUBE GROUP

MEET THE RUBIK'S CUBE

- Best-selling toy in the world – 500 million sold as of Jan 2024
- Invented in 1974 by Hungarian architect Erno Rubik

THE STRUCTURE OF THE RUBIK'S CUBE

In terms of faces:

- **6** faces
- **54** 'facelets' i.e. 9 on each face

In term of 'cubies':

- **26** cubies
- **12** edges – two colours
- **8** corners – three colours
- **6** centres – one colour

Both views are useful!

WHY A GROUP?

A **group** is a pair (G, *) which satisfies the following:

(G1) G is a **non-empty** set

(G2) * is an **associative binary** operation

(G3) G has an **identity element**

(G4) **Every** element of G has an **inverse in G**

Translation:

G has something in it

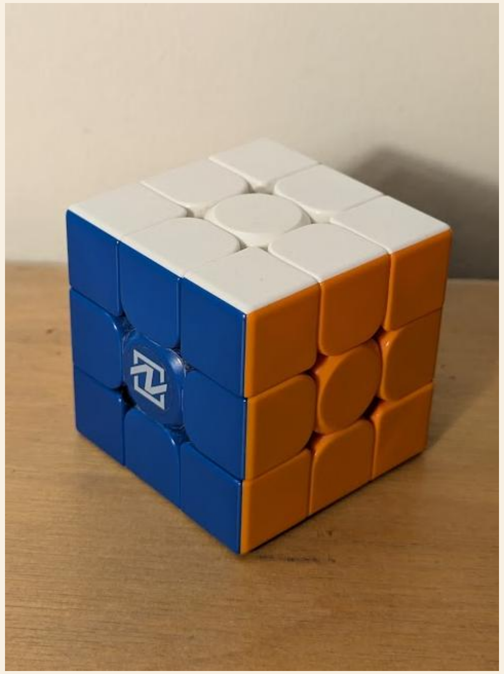
$(a*b)*c = a*(b*c)$

$e*a = a$, for all a

$a*b = e$, for some b

THE RUBIK'S CUBE GROUP

The Rubik's cube forms a group with the **permutations** that reach its possible states, under **composition**.



G AS A SUBGROUP OF S_N

48 movable facelets – so 48! ways of arranging them?

$48! = 1.24 \times 10^{61}$ (roughly)

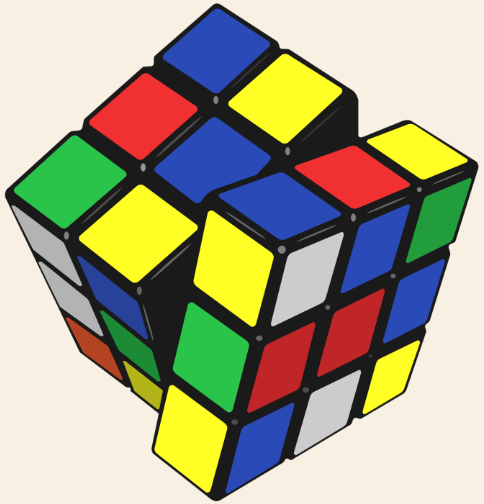
This is **structure** is similar to a standard group:

- **S_n** – The symmetric group on **n things**
- Form of group of **bijective permutations**, under **composition**

$$G \subseteq S_{48}$$

CONSTRAINTS, CUBIES, AND PARITY

- Rubik's cube has **physical** constraints
- Rubik's cube group inherits these as **algebraic** constraints



Physical constraint:

Cubies can only be permuted by **rotating a face**.

Algebraic constraint:

Only permutations with **even parity** are allowed.

i.e. permutations which are an **even** number of **transpositions** (2-cycles)

Turning a face is equivalent to **two 4-cycles**, which in turn make 6 transpositions.

This is 'modelled' by the **alternating group** A_n of even permutations of n things, under composition.

EDGE ORIENTATION CONSTRAINTS

- Edges of the cube have two colours and thus **two orientation states**.

- Call these states:

  0 (correct)

  1 (flipped)

Physical constraint:

Rotating a face always changes the **orientation** of edges in a **structured** way.

Algebraic constraint:

The **sum** of the orientation states of edges is always **0 mod 2**,

i.e. the sum is a multiple of 2.

This is 'modelled' by $\mathbb{Z}_2$, the group of **integers with addition mod 2**

CORNER ORIENTATION CONSTRAINTS

- Corners of the cube have three colours and thus **three orientation states**.
- Call these states:
  0 (correct)
  1 (anti-clockwise)
  2 (clockwise)

Similarly to the edges, this results in the **algebraic constraint** that the **sum** of the orientation states of corners is always **0 mod 3**,

i.e. the sum is a multiple of 3.

This is 'modelled' by $\mathbb{Z}_3$, the group of **integers with addition mod 3**.

ISOMORPHISM AND THE SEMIDIRECT PRODUCT

When we say ‘modelled’, we mean **isomorphic**
i.e. there exists a **bijective** function between the group elements which **preserves** the group’s **structure**. This is denoted as $\boldsymbol{X} \cong \boldsymbol{Y}$.

The Rubik’s cube group is isomorphic to:

(Orientation group) $\rtimes$ (Permutation group)

The operation $\rtimes$ is the ‘semidirect’ product, which can be thought of in simple terms as a product which encodes how changing one group affects the other i.e. how one group rearranges the elements of another (for our purpose).

PUTTING IT ALL TOGETHER

$$G \cong (\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2)$$

- 8 corners, minus 1 constraint ($\mathbb{Z}_3^7$)
- 12 edges, minus 1 constraint ($\mathbb{Z}_2^{11}$)
- Permutation affects orientation ($\rtimes$)
- Only even parity permutations allowed ($A_8 \times A_{12}$)
- Parity flips are paired ($\mathbb{Z}_2$)

$$|G| = 3^7 \times 2^{11} \times \frac{8!}{2} \times \frac{12!}{2} \times 2 = 43{,}252{,}003{,}274{,}489{,}856{,}000 \qquad (\sim 10^{19})$$

MASSIVE!

THANK YOU AND QUESTIONS

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